university of groningen / faculty of science and engineering

Control Engineering for BME

Final Exam

15:00-18:00*, Friday 5^{th} of November, 2021,
Academic year 2021/2022 1A

Teacher: Rodolfo Reyes-Báez

The exam will *focus on feedback control systems design (state-feedback, observer-based output feedback, PID controller) and their closed-loop performance analysis (Routh-Hurwitz, steady-state, root-locus, Bode plots and Nyquist criterion).*

Please, read carefully the following information.

- Please write completely your name, student ID number and *study program.* Please, write your answer using a pen, not a pencil.
- The exam contains four questions with compulsory sub-questions and an optional sub-question for which extra points can be granted (if correct).
- For every question, write your answer neatly on blank papers and write down your name, student number and study program on the top of each page.
- This is a closed-book exam, but you can use calculator. Please write down your answer clearly and with proper argumentation/reasoning whenever needed. Providing only the final answers without proper argumentation is **not** acceptable[1].
- If you are unclear about a specific problem, you can make your own assumptions. Describe your assumptions at the beginning of your answer. **Keep in mind that if the assumptions are no correct, your solution will not be ether.**
- Whenever we think is appropriate, a follow-up **oral** examination to suspected cases will be arranged before the final grade is determined. In this case, the follow-up oral examination will be based on the questions of this exam and the final grade will be based on the same weighting factor as before where the adjusted grade from the oral examination will be used instead that replaces the final exam grade.
- If you return the sheets, then your exam will be graded, unless you explicitly write "do not grade" on the first page. If the exam is graded, it will be registered.

For the grader only

	Exercise 1	**Exercise 2**	**Exercise 3**	**Exercise 4**
Points				
Bonus				

*Except for those who have the right to have 30 minutes extra due to medical reasons.

[1] Concepts over computations, but both are important.

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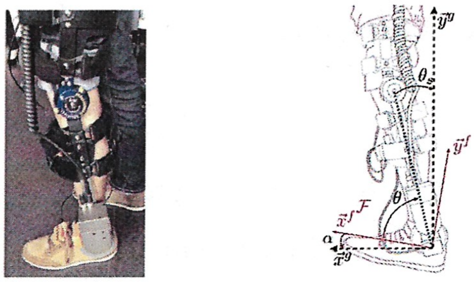
Question 1: Stabilization of a nonlinear system via state feedback (10pts)

Consider the actuated-ankle-foot orthosis (AAFO) shown in Figure 1.



(a) (b)

Figure 1: (*a*) Actuated ankle-foot orthosis and (*b*) its engineering schematic

From a control-oriented modeling perspective, in some applications where the environment conditions are *ideal*, like in rehabilitation, some physical phenomena (friction, lateral forces, etc.) can be neglected, and the AAFO can be modelled as an balance system (inverted pendulum). By invoking first principles (Newton's laws or Euler-Lagrange equations), one gets the following *nonlinear* state space dynamic model with state variables $x_1 = \theta$ and $x_2 = \dot{\theta}$:

$$\dot{x} = \begin{bmatrix} x_2 \\ \frac{mgl}{J_t}\sin(x_1) - \frac{b}{J_t}x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{l}{J_t}\cos(x_1) \end{bmatrix} u, \qquad (1)$$

$$y = x_1,$$

where m is the mass, J_t is the total inertia of the AAFO, g the gravity constant, l is the distance from the origin to the enter of mass, b the damping coefficient, and u the input. The output $y = x_1$ is the position of the AAFO's actuator, see Figure 1 (*a*). For this exercise, take $b = 0.2$, $J_t = 0.1$, $l = 0.5$, $m = 2$ and $g = 9.81$.

1. **(3pts)** Find the linear approximation of (1) around the operation point

$$(\bar{x}, \bar{u}) = \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, 0 \right). \qquad (2)$$

   That is, compute matrices A, B, C, D of the linearized system given by

$$\begin{aligned} \delta\dot{x} &= A\delta x + B\delta u, \\ \delta y &= C\delta x + D\delta u, \end{aligned} \qquad (3)$$

   where $\delta x = x - \bar{x}$, $\delta u = u - \bar{u}$ and $\delta y = y - \bar{y}$, with $\bar{y} = C\bar{x} + D\bar{u}$.

2. **(3pts)** Compute the reachability matrix W_r for the linearized system in (3), and discuss whether or not the system is reachable. If it is reachable, write the reachability canonical form, that is, matrices $\tilde{A}$ and $\tilde{B}$.
   Hint: If you did not compute A, B, C, D in the previous sub-problem, use the following

$$A = \begin{bmatrix} 0 & 1 \\ \frac{mgl}{J_t} & -\frac{b}{J_t} \end{bmatrix}, B = \begin{bmatrix} 0 \\ \frac{1}{J_t} \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}, D = 0. \qquad (4)$$

3. **(4pts)** If the system is reachable, design the matrix gain $K = [k_1\ k_2]$ in the state feedback control $\delta u = -K\delta x$ such that the closed-loop system's matrix $A_{\text{cl}} = A - BK$ has its eigenvalues at $s = -1$ and $s = -2$.
   Hint: Use the matching target polynomial method or the reachability canonical form method.

4. **Bonus (3pts).** Consider the original *nonlinear* model of the balance system in (1) in closed-loop with the control $u = \bar{u} - K(x + \bar{x}) = -Kx$:

$$\dot{x} = \begin{bmatrix} x_2 \\ \frac{mgl}{J_t}\sin(x_1) - \frac{b}{J_t}x_2 - \frac{l}{J_t}\cos(x_1)k_1 - \frac{l}{J_t}\cos(x_1)k_2 \end{bmatrix}, \qquad (5)$$
$$y = x_1.$$

Determine the linearized system of (5) around the equilibrium point $\bar{x} = (0, 0)$ and check if the linearized system's matrix is Hurwitz. What can you say about the stability of the equilibrium point $\bar{x}$ of the closed-loop nonlinear system in (5)? Explain in one sentence.
Hint: If you did not find K in Exercise 1.3, then use $K = [(2g + 0.4)\ 0.2]$.

Question 2: Observer-based output linear feedback for the AAFO (10pts)

Consider the linearized system that you obtained in Question 1.2 (or the one in the hint).

1. **(1pts)** Determine the observability matrix W_o and discuss if the system is observable.
2. **(2pts)** Determine the observable canonical form of the system, i.e., matrices $\tilde{A}$ and $\tilde{C}$.
3. **(4pts)** Determine the gain matrix L such that the eigenvalues of $(A - LC)$ are located at $s = -1$. Write explicitly the observer for the linearized system:

$$\delta\dot{\hat{x}} = A\delta\hat{x} + B\delta u + L(\delta y - C\delta\hat{x}), \qquad (6)$$

where $\delta\hat{x}$ is the observer estate which estimates the state of the linearized system δx.

4. **(3pts)** Using the gain matrix K of Question 1.3 and the gain matrix L of Question 2.3, determine an observer-based dynamic output feedback controller that solves the output regulation problem

   (a) the closed-loop system is asymptotically stable

   (b) the output δy asymptotically convergences to the constant reference signal $r = 0$.

   Hint: If you did not find K and L, then use $K = [(2g + 0.4)\ 0.2], L = [0\ 10g + 1]$.

5. **Bonus (2pts).** From a practical control engineering perspective, are the eigenvalues of the closed-loop system and observer properly assigned? Discuss it in two lines.
Hint: use the concept of dominant eigenvalues for the overall closed-loop system.

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Question 3: Frequency modeling and PID control of a DC motor (10pts)

1. **(4pts)** The Bode plot associated to the frequency response of a DC motor is depicted in Figure 2, which is obtained from input/output measurements. Determine the motor transfer function $P(s)$ from the Bode plot.

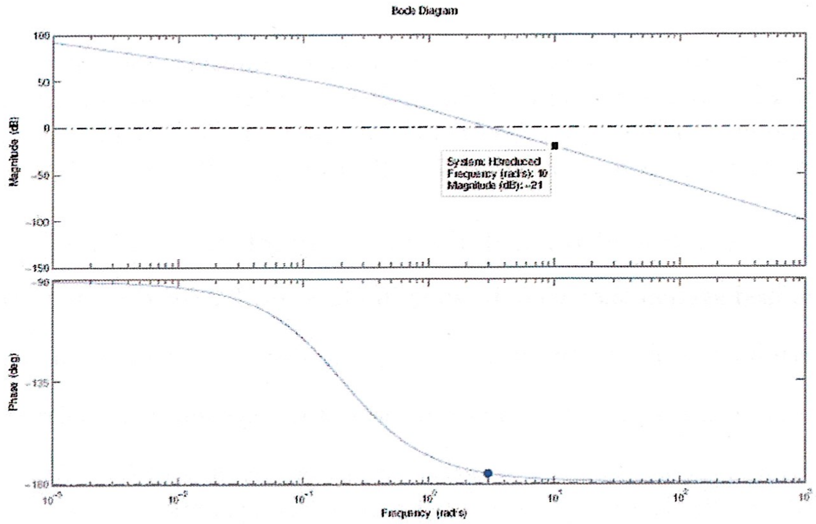

Figure 2: Bode plot of a DC motor, the indicated point is $20\log_{10}|L(j10)| = -21$.

Hint: Note that the phase starts at $-90°$ *due to a pole at* $s = 0$, *and then drops to* $-180°$.

2. **(6pts)** Consider transfer function $P(s)$ of Question 3.1 in closed-loop with the PID controller

$$u = K_p e(t) + K_i \int_0^t e(\tau)d\tau + K_d \frac{de}{dt}(t) = K_p\left(e(t) + \frac{1}{T_i}\int_0^t e(\tau)d\tau + T_d\frac{de}{dt}(t)\right)$$

with $T_i = K_p/K_i$ the reset time, and $T_d = K_d/K_p$ derivative time. Figure 3 shows the block diagram of the the closed-loop feedback control system. Suppose that the PID controller gains K_p, K_i, K_d are chosen such that the steady-state output $y_{ss} \to r$ when $d = 0$. For $d \neq 0$, calculate the transfer function from the constant disturbance d to the output y. Does the PID controller yield a steady-state error with respect to the disturbance d? Is there a steady-state error if we choose a PD controller ($K_i = 0$) instead?

Hint 1: Use the final value theorem to analyse the steady-state y_{ss} *due to the input* d.
Hint 2: If you did not find $P(s)$ *in 3.1, then use* $P(s) = \frac{\alpha}{s(s+0.2)}$, *with* $\alpha = 8.9143$.

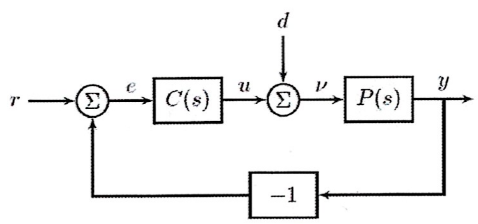

Figure 3: Block diagram of a feedback system with reference r, disturbances d, and output y.

3. **Bonus (3pts)**. Change the plant by

$$P(s) = \frac{1}{(s+1)^3}. \tag{7}$$

Design the gains k_p, T_i, T_d for the P, PI and PID controllers using Ziegler-Nichols method based on the frequency response.

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Question 4: Frequency domain analysis and design of feedback system (10pts)

Consider a process transfer function $P(s)$ with a proportional controller $K > 0$ and unity feedback depicted below in Figure 4.

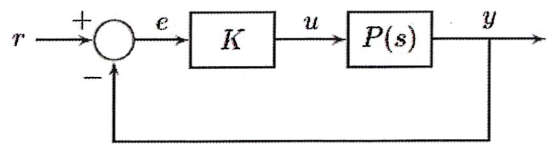

Figure 4: Feedback system with a proportional controller

1. **(1pts)** Determine the loop transfer function $L(s)$.
2. **(1pts)** Take the process transfer function as

$$P(s) = \frac{s+4}{(s+2)^2(s-1)}. \tag{8}$$

   Is the open-loop system stable?

3. **(4pts)** The Bode plot and Nyquist plot of the loop transfer function $L(s)$ for $K = 2$ are given as shown in Figure 5 and Figure 6. Indicate the gain margin and phase margin in both plots. Use the Nyquist criterion to determine if the closed-loop system is stable.

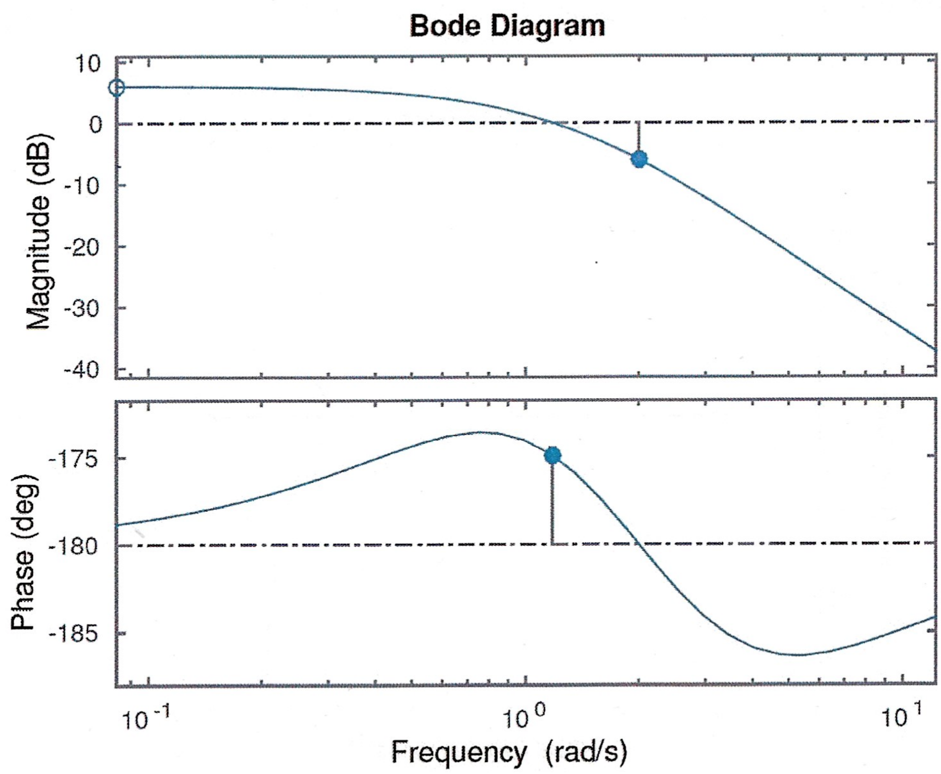

Figure 5: Bode plot

4. **(4pts)** How should you change K to achieve a phase margin of $\varphi_m = 6°$?
   Hint: The phase at $\omega \approx 0.77$ *is* $\angle L(j\omega) = -174°$.

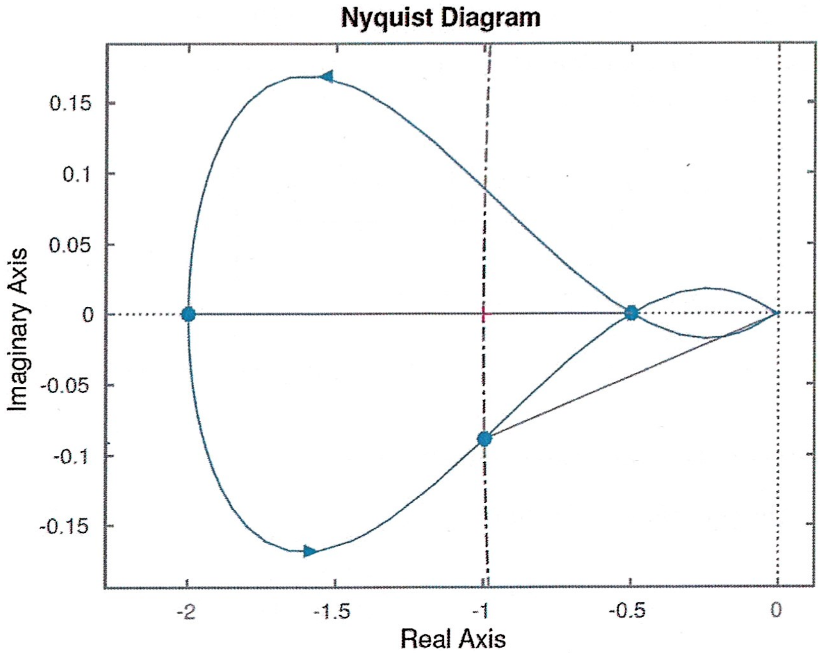

Figure 6: Nyquist plot

5. Bonus (**2pts**). The root locus of the feedback system in Figure 4 is shown in Figure 7. Is it possible to place a pole of the feedback system at $s = -5$ by varying the proportional gain K? Why there are two branches that go to infinity? Motivate your answer.

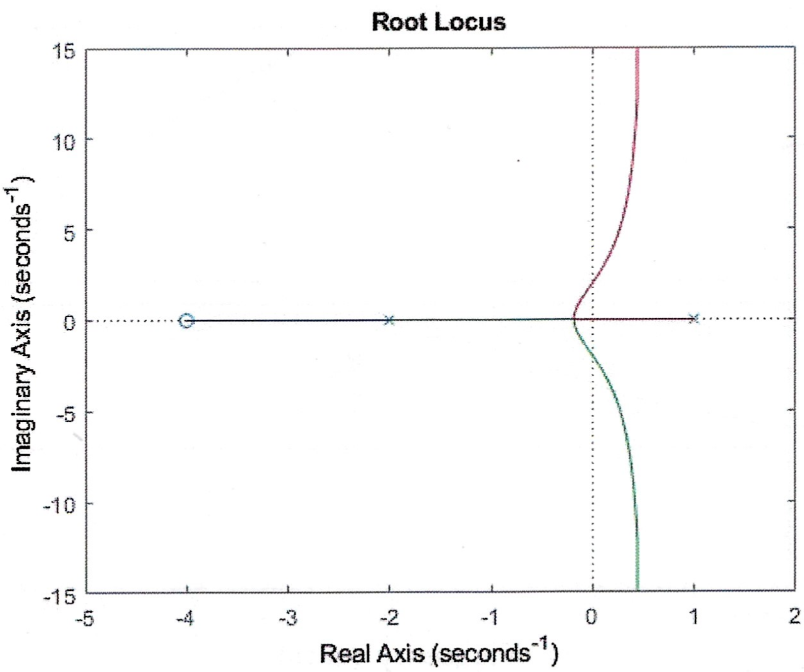

Figure 7: Root locus of $L(s) = KP(s)$.

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Formulas

Jacobian linear approximation

For the system $\dot{x} = F(x,u)$, with $x \in \mathbb{R}^2$ and $u \in \mathbb{R}$, the output $y = h(x,u) \in \mathbb{R}$, and the operation point $(\overline{x}, \overline{u})$, the Jacobian linearization can be computed as follows

$$A = \frac{\partial F}{\partial x}(x,u)\big|_{(\overline{x},\overline{u})} = \begin{bmatrix} \frac{\partial F_1}{\partial x_1}(\overline{x},\overline{u}) & \frac{\partial F_1}{\partial x_2}(\overline{x},\overline{u}) \\ \frac{\partial F_2}{\partial x_1}(\overline{x},\overline{u}) & \frac{\partial F_2}{\partial x_2}(\overline{x},\overline{u}) \end{bmatrix}, \quad B = \frac{\partial F}{\partial u}(x,u)\big|_{(\overline{x},\overline{u})} = \begin{bmatrix} \frac{\partial F_1}{\partial u}(\overline{x},\overline{u}) \\ \frac{\partial F_2}{\partial u}(\overline{x},\overline{u}) \end{bmatrix},$$

$$C = \frac{\partial h}{\partial x}(x,u)\big|_{(\overline{x},\overline{u})} = \begin{bmatrix} \frac{\partial h}{\partial x_1}(\overline{x},\overline{u}) & \frac{\partial h}{\partial x_2}(\overline{x},\overline{u}) \end{bmatrix}, \quad D = \frac{\partial h}{\partial u}(x,u)\big|_{(\overline{x},\overline{u})}.$$

Characteristic polynomial

Consider the LTI system

$$\dot{x} = Ax + Bu, \ y = Cx. \tag{9}$$

with $x \in \mathbb{R}^2, u \in \mathbb{R}$, and $y \in \mathbb{R}$. The characteristic polynomial of the system matrix is

$$p(s) = \det(sI - A) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n. \tag{10}$$

Reachability matrix

For the system in (9), the reachability matrix is

$$W_r := [B \mid AB \mid A^2B \mid \ \cdots \ \mid A^{n-1}B].$$

Reachability Canonial form

If (9) is reachable, then its associated reachable canonical form is $\dot{z} = \tilde{A}z + \tilde{B}u$, with matrices

$$\tilde{A} = \begin{bmatrix} -a_1 & \cdots & -a_{n-1} & -a_n \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}, \qquad \tilde{B} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{11}$$

Note: The characteristic polynomials of system $\dot{x} = Ax + Bu$ and $\dot{z} = \tilde{A}z + \tilde{B}u$ are the same.

Output regulation problem via full-state feedback control

Problem: Given a reference r for the LTI system in (9), design u such that $y \to r$ as $t \to \infty$.

Solution: The following full-state feedback solves the output regulation problem

$$u = -Kx + k_r r, \tag{12}$$

where the the *feedback gain* K is such that the matrix $A_{\text{cl}} := (A - BK)$ is Hurwitz; and the *feedforward gain* is chosen as

$$k_r = -\frac{1}{C(A-BK)^{-1}B}.$$

Full state feedback control design via eigenvalue assignment

If system (9) can be put into the reachable canonical form (11), then the controller in (12) is

$$u = -\tilde{K}Tx + k_r r = -\left[(\alpha_1 - a_1) \quad (\alpha_2 - a_2) \quad \cdots \quad (\alpha_n - a_n)\right] \tilde{W}_r W_r^{-1} x + k_r r$$

where the n-desired eigenvalues that are solution to the target characteristic polynomial

$$p_{\text{tg}}(s) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1}s + \alpha_n, \tag{13}$$

with $\tilde{W}_r$ the reachability matrix of the system in reachable canonical form.

Observability matrix

The observability matrix of system (9) is

$$W_o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}. \tag{14}$$

Observability canonical form

The observable canonical form of the system in (9) with characteristic polynomial (10) is

$$\begin{aligned} \dot{z} &= \underbrace{\begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 & \\ -a_2 & 0 & 1 & \cdots & 0 & \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ -a_{n-1} & 0 & 0 & \cdots & 1 & \\ -a_n & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}}_{\tilde{A}} z + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix} u, \\ y &= \underbrace{\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}}_{\tilde{C}} z + Du. \end{aligned} \tag{15}$$

Luenberger observer design

Consider the system in (9) with characteristic polynomial (10). The the dynamical system

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}), \tag{16}$$

is an observer for the system, with L a $(n \times 1)$ gain matrix designed such that

$$p_{\text{tg}}^{\text{obs}}(s) = \det(sI - (A - LC)) = s^n + \beta_1 s^{n-1} + \cdots + \beta_{n-1}s + \beta_n = 0 \tag{17}$$

has all the roots with strictly negative real parts. The designer can decide the location of observer eigenvalues that are solutions to the observer target characteristic polynomial in (17).

Luenberger observer design via eigenvalue assignment

If system (9) can be put into form (15), then the $(n \times 1)$ gain matrix L in (16) can be chosen as

$$L = T_o^{-1}\tilde{L} = W_o^{-1}\tilde{W}_o \begin{bmatrix} \beta_1 - a_1 \\ \beta_2 - a_2 \\ \vdots \\ \beta_n - a_n \end{bmatrix}.$$

Observer-based dynamic output feedback

Consider the system in (9). The observer-based dynamic controller

$$\begin{aligned} \dot{\hat{x}} &= (A - BK - LC)\hat{x} + Ly + Bk_r r \\ u &= -K\hat{x} + k_r r \end{aligned}$$

yields the closed-loop characteristic polynomial

$$p_{\text{tg}}^{\text{dyn}}(s) = p_{\text{tg}}(s) \cdot p_{\text{tg}}^{\text{obs}}(s).$$

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Input/output response to exponential input

$$y(t) = Ce^{At}x(0) - Ce^{At}(sI - A)^{-1}B + \left[C(sI - A)^{-1}B + D\right]e^{st}$$

Bode factors type

Consider the frequency response of the loop transfer function $L(s)$ in Bode form, i.e.,

$$L(j\omega) = KP(s) = K_0\,(j\omega)^{-1}\left(\frac{j\omega}{a} + 1\right)^{-1}. \tag{18}$$

The magnitude and phase of each factor are

1. Type 1: $K_0(j\omega)^{-1}$
   (a) Magnitude in dB: $20\log_{10}|K_0(j\omega)^{-1}| = 20\log_{10}|K_0| - 20\log_{10}\omega$.
   (b) Phase: $\angle K_0(j\omega)^{-1} = \angle(j\omega)^{-1} = \angle - (j\omega) = -90°$.
2. Type 2: $(j\omega\tau + 1)^{-1}$
   (a) Magnitude in dB: $20\log_{10} M(\omega) = -20\log_{10}|(j\omega\tau + 1)|$.
   (b) Phase: $\angle(j\omega\tau + 1)^{-1} = -\arctan(\omega\tau)$

Final value theorem

If $\lim_{t\to\infty} f(t)$ has a finite limit, then $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$, where $F(s) = \mathcal{L}\{f(t)\}$.

PID control tuning via the frequency response-based Ziegler-Nichols method

Suppose that a PID controller is connected to the process $P(s)$, and the integral and derivative actions are set to zero. The gain is increased until the system starts to oscillate. The critical value of the proportional gain k_c is observed together with the period of oscillation T_c at frequency $\omega_c = 2\pi/T_c$. The closed-loop system becomes marginally stable. Then, the suggested controllers gains are given in the following table:

Type	k_p	T_i	T_d
P	$0.5k_c$		
PI	$0.4k_c$	$0.8T_c$	
PID	$0.6k_c$	$0.5T_c$	$0.125T_c$

Relative stability based on margins

The critical gain and phase conditions $|KP(j\omega_{\text{pc}})| = 1$, and $\angle P(j\omega_{\text{pc}}) = 180°$

- Gain margin (g_m): is the factor by which K can be multiplied before $|KP(j\omega)| = 1$ when the phase $\angle P(j\omega_{\text{pc}}) = 180°$; that is, $g_m = 1/|L(j\omega_{\text{pc}})|$.
- Phase margin (φ_m): is the amount by which the phase at the crossover frequency ω_{gc} differs from 180° mod 360°; that is, $\varphi_m = 180° + \angle L(j\omega_{\text{gc}})$.

Nyquist stability criterion

Let $L(s)$ be the loop transfer function for a unity negative feedback system. Assume that $L(s)$ has P poles enclosed in the Nyquist contour Γ. Let N be the number of **clockwise** encirclements of $(-1, j0)$ by the Nyquist plot of $L(s)$ when s moves along Γ in **clockwise** direction. Then the closed-loop system has $Z = N + P$ poles in the **right-half plane**.